THE COMPREHENSIVE EXAMINATION IN THE TEACHING OF SECONDARY MATHEMATICS.

By Howard F. Hart.

The problem of this paper is to trace the changes in the subject matter and method of secondary mathematics which would seem to be required by an examination which is to test the student's ability to use his mathematics as tools to accomplish the tasks set him.

It is not a part of this problem to consider whether or not such examinations can be written, to consider whether or not it is possible for a student to cram for such examinaions, to consider whether or not it is justifiable to set examinations in which a half or over of the candidates may fail. Nor is it, furthermore, a part of the problem to consider the specific form of such an examination. Nevertheless it is pertinent for us to observe that all the questions of such a test must be such that their solution does not depend upon the use of tricks or special devices but upon the intelligent use of the regular general methods.

Finally, it is not my purpose to consider any questions which have to do mainly with the related problem of the curriculum.

The general conclusions seem to follow at once from the definition of the terms involved and the limitations of the problem. I believe that the comprehensive examination requires the simplification of the subject matter of mathematics by the elimination of needless repetition, the selection of the best methods of procedure where competing methods exist, and the modernization of certain ancient methods of treatment whose only recommendation seems to be their antiquity. Of more importance is the necessity of the selection of methods of teaching which will develop in the student initiative, independence, and power. Of the highest importance is the change which is necessary in the point of view of the teacher. There seems to be no way to establish these conclusions. We can only make their necessity and meaning clear by concrete illustrations.

One illustration of waste through needless repetition is found

in the more or less complete reproof of previous theorems in a given geometrical proof. This waste is more serious than it would seem to be. It is a common fault in textbooks and can only be obviated by not taking the given proof on faith. The following are three theorems whose common proof is typically faulty in this respect: (1) A line perpendicular to each of two intersecting lines is perpendicular to their plane. (2) If a line intersects a plane, the line in the plane perpendicular to the projection of the first line at the point of intersection, is perpendicular to the line itself. (3) Parallel transverse sections of a pyramidal space are similar polygons whose areas are proportional to the squares of their distances from the vertex.

The present treatment of imaginaries is an illustration of waste from the same point of view. If the imaginary were treated throughout as the product of a real quantity and the unit i the necessity for a separate treatment disappears. It should become a part of a chapter on exponents.

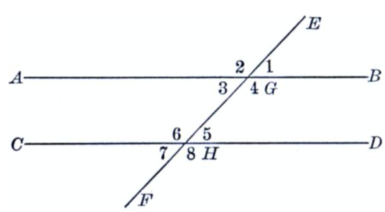
Perhaps the best illustration of waste through the development of competing methods is found in radicals and exponents. The radical sign and its theory are not necessary. I am in favor of dropping them altogether. The time saved would be considerable and would enable us to put more time where it is very much needed, *e. g.*, inequalities.

In considering the modernization of the subject matter one is simply appalled at the prospect. The necessity for it seems to be everywhere. And so I can only hope to call attention to one or two typical examples. First, perhaps, there is that ancient humbug, proportion. When we realize that the writing of it in the common form, which hides its true nature, is the only necessity for most of the terms which we use in it, we should be willing to write it in the form $A/B = C/D$ and to drop that mess of ancient verbiage "antecedent," "consequent," "mean," "extreme," etc., and in time (I almost despair of it) we might find some acceptable substitutes for its remaining ambiguous nomenclature. Proportion is very valuable as a method of reduction of identities and equations but there should not be any theory of proportion as such. The student should know merely, with reasons why, that whenever two fractions are identically equal, certain pairs of derived fractions are also equal.

A close second to proportion is the theory of exponents with its proofs which are not proofs. When we are only seeking to find the *meaning* of certain exponential forms under the principle of *no exception* why should we not frankly say so. It is better mathematics, it is actually intelligible to the average student, and it takes less time.

And then there are the equations. How much longer are we going to apply axioms that do not apply? For a clear concise account of why they do not apply and for a common-sense treatment of the equivalency of equations I refer you to Dr. J. M. Taylor's article in *The Mathematics Teacher* for June, 1910. After reading his paper you will be prepared to enjoy some of our recent texts.

A typical illustration from geometry is the present grouping of the angles formed when two lines are cut by a transversal. Thus if in the figure we consider all angles to be rotated contra-



clockwise and introduce the ideas of initial and terminal arms then the eight angles can be divided into two sets of four each: (1, 3, 5, 7) and (2, 4, 6, 8); since the transversal is a common initial and terminal arm for the set. Usually whatever is true of one angle is true of the set. Thus, *e. g.,* "If two non-vertical angles in one set are equal, then all the angles in each set are equal and the angles of one set are the supplements of the angles in the other," or again, "If two non-adjacent angles in different sets are supplemental the same conclusions follow." By using the ideas of these sets we are able to state the theory of parallels in but four theorems.

I have said that it is of more importance that the methods of teaching should be such as to produce in the highest possible measure initiative, independence, and power. I most emphatically dissent from the idea that the methods of presenting a subject are of minor importance. They are of fundamental importance. The subject matter is mere material. The value of

the result depends upon the methods. For example, I believe absolutely that the reason geometry has been so much attacked is the failure to make it yield initiative, independence, and power by the traditional method. Geometry is ideal material for such purposes and I believe just as firmly that it will yield results when presented by the right methods.

In general, it seems to me that any method must satisfy the following requirements: (1) Anything which is acquired without the use of the reasoning powers has little educational value.* (2) Progress must be through the medium of clear ideas from the known to the related unknown. (3) Initiative, independence, and power can not be taught, they must be acquired through activity. Therefore it seems to me that the methods must be actual rather than philosophic and inductive rather than deductive. By all of which I mean that the student should be given the materials of the subject, just as he is given lumber and metal in the shop, and beginning with their simplest combinations be led to do for himself a list of things selected with regard to the time available, their relation to each other, and their intrinsic value.

In order to show one application of these ideas I have brought along the first lessons of an outline for the beginning of geometry.

Lesson I. Place different kinds of lines (except straight line) on the board and work out the test. Then give the straight line and lead to the definition of line, sect, and point. Straight line axiom, broken line, and curved line.

Home work: Addition and subtraction of sects.

Lesson II. Review: Recall generalizations of previous lesson.

Main topic: Place different polygons on the board and derive classifications (as to number of sides and form). Give usual names.

Home work: Have polygons constructed.

Lesson III. Review: Lessons I and II.

Main topic: Ask class for ideas of an angle. Tabulate and compare. Parts of an angle. Reading an angle, initial and terminal arms.

Home work: Assign intersecting lines and polygons with vertices lettered contra-clockwise, and ask upils to read the angles contra-clockwise.

Lesson IV. Review: Kinds of lines; straight, broken, and curved. Perimeter of polygon as sum of sects.

Main topic: The angles of intersecting lines classified as adjacent,

* Cf. Professor Dewey's "How We Think."

straight, supplementary, vertical, perigon, explementary. Point out how angles are added, infer subtraction.

Home work: Assign polygons with sides produced and have angles (interior, exterior, etc.) read. Draw figures for equal adjacent supplements and explements.

Lesson V. Review: Definitions of classes of angles. Explement and supplement. Give name to right angle of exercise of last lesson.

Main topic: Tests of geometric equality.

(*a*) Senses not reliable in close decisions.

(*b*) Coincidence. Application in "All straight angles are equal."

Home work: A list of axioms of equals. A list of theorems of unequals.

Lesson VI. Review: Lesson V in general.

Main topic: (*c*) Axioms of equals. Application in "A sect has but one midpoint," "An angle has but one bisector," and "All right angles are equal."

Home work: Prove "A sect has but one trisection point from either end" and "All perigons are equal."

Lesson VII. Review: Sum of angles. Complement and perpendicular. Straight line axiom.

Main topic: Coincidence from the point of view of determination.

(*a*) Lines: i. Straight line axiom.

ii. Bisection of an angle.

iii. Perpendicular at a point. Proof inferred as a special case.

(*b*) Point. By two intersecting lines. Proof by the straight line axiom.

Home work: Draw figures for and write out the steps of the previous theorems.

.... etc. ....

(This outline was prepared by Mr. William Brubaker, Mr. Elmer F. Conine, and the author, for use in their work in the Montclair high school.) While these lessons are under way no textbook is used. At the close of this work one is furnished in which the material covered here is given in such a way that it can be consulted as a reference. For the remainder of the course practically no readymade solutions are given. The students are led to find their own and having found them, to preserve them in a notebook for use in formal recitations, summaries and reviews, and for future reference.

I have said that a change is required in the teacher. All will agree, I think, that if any change is required it is a change of the highest importance, for the teacher is and must be the chief

factor in secondary education. I do not propose to discuss whether or not a change is required beyond suggesting to any one interested a study of the reports of the various examining bodies, especially the reports of the College Entrance Examination Board, and to repeat a statement made to me within a month by the salesmanager of one of our largest book companies. The statement was to the effect that the book companies found it very difficult to sell any new geometries because the teachers having once become acquainted with the order of propositions and the solutions of the originals of one book would not consider one having a different order or different exercises.

However, if we assume that a change is required then it seems to me that it must lie in our point of view of our relation to the possibility of producing a product able to be tested by a comprehensive examination. I believe that it is being done now and I also believe that it can be done by any teacher who has had sufficient training and uses proper methods. But it must be realized once for all that power can not be taught. It can only come through development gained by self-activity. Under no circumstances can we "tell" into a boy the ability to solve an original geometrical exercise. And to learn and recite the steps of any number of solutions thought out by some one else will not do either. If this seems to be a strong statement we have but to reflect upon the usual horror of students for geometrical originals and their ready ability and interest in algebraic originals. While these exercises are different in kind, and allowance must be made for that, nevertheless the ultimate reason lies in the difference of the ways in which we have attacked the two classes of exercises. As Euclid once said so must we also say that there is "no royal road" to power except through the actual doing of the things over which power is required.

Finally, we must come to appreciate the results of an indifferent stand-pat attitude toward our work. If we are content to go on doing just the same things in the same ways with no thought or question of improvement in time the possibility of change will be beyond us. Let us remember that, "Unto every one that hath shall be given; but from him that hath not, even that which he hath shall be taken away from him."

HIGH SCHOOL,
MONTCLAIR, N. J.